Lehigh University
Lehigh Preserve

Fritz Laboratory Reports

Civil and Environmental Engineering

1968

The plastic indentation of metal block by a flat punch, September 1968 (70-12)

W. F. Chen.

LEHIGH UNIVERSITY INSTITUTE OF RESEARCH

Solid Mechanics, Plasticity, and Limit Analysis

THE PLASTIC INDENTATION OF METAL BLOCKS BY A FLAT PUNCH

by
Wai-Fah Chen

October, 1968

Fritz Engineering Laboratory Report No. 356.4

Solid Mechanics, Plasticity, and Limit Analysis

THE PLASTIC INDENTATION OF METAL BLOCKS BY A FLAT PUNCH

by

Wai-Fah Chen

National Science Foundation
Grant Gk-3245 to Lehigh University

Fritz Engineering Laboratory
Department of Civil Engineering
Lehigh University
Bethlehem, Pennsylvania

October, 1968

Fritz Engineering Laboratory Report No. 356.4

THE PLASTIC INDENTATION OF METAL BLOCKS BY A FLAT PUNCH

Wai-Fah Chen*

ABSTRACT

Limit analysis is utilized to obtain upper and lower bounds of the punch pressure during indentation of a square punch on a square block, and a circular punch on a circular cylinder.

Reasonably close upper and lower bounds for the average pressure over the square and circular punches are obtained, and are expressed as functions of the ratio of block width to punch width.

*Assistant Professor, Fritz Engineering Laboratory, Department of Civil Engineering, Lehigh University, Bethlehem, Pennsylvania.

TABLE OF CONTENTS

1. INTRODUCTION

Upper and lower bounds for the indentation pressure have been previously obtained for the indentation of an infinitely large block by a flat-ended rigid punch. For a square punch and Tresca yield condition, it was shown that the average collapse pressure for smooth and rough contact surfaces lies between the values 5k and 5.71k, where k is the maximum shearing stress permissilbe in the body[4]. For the case of axial symmetry, the corresponding value is 5.69k or 6.05k depending on whether the punch is smooth or perfectly rough[3,5]. However, the problem of indentation on a block of finite dimensions has not been attempted to date. This is a relevant problem for the notched bar in tension, which is mathematically equivalent to a punch in compression.

The following work is essentially a continuation of those papers referenced above. Here, as in Reference 4, limit analysis is used to obtain upper and lower bounds of the limit punch pressure during indentation of a square punch on a square block, and a circular punch on a circular cylinder. The material of the block and cylinder is assumed to be an elastic-perfectly plastic material which obeys the Tresca yield condition and the associated flow rule.

2. SQUARE AND CIRCULAR PUNCHES - LOWER BOUNDS

To obtain a lower bound for the average indentation pressure, the discontinuous stress pattern in Fig. 1 (Winzer and Carrier[6,7]) is found to be useful for extension into three dimensions. The pressure on A-B has the value $2k(1 + \sin\alpha)$ when the angle of the wedge is 2α. The region vertically below D-E is in a state of uniaxial compression of magnitude $2k(1 - \sin\alpha)$.

The corresponding three-dimensional stress field for the square punch is shown in Fig. 2. L-M-N-O is the square area of punch. The triangular region A-B-C in Fig. 1 becomes the pyramid-shape volume L-M-N-O-C. The four triangular faces of the volume are all inclined at an angle of γ to the vertical. The triangular regions B-C-E and C-E-F in Fig. 1 become the volumes M-N-S-R-C and R-S-C-F respectively. The line R-S is parallel to the sides M-N and L-O of the square punch, and the rectangular face M-N-S-R and triangular face R-S-C of the volume M-N-S-R-C are inclined at an angle of α and γ' to the vertical respectively.

The pressure $2k(1 + \sin\alpha)$ on the square area is supported by uniaxial compression of amount $2k$ in the four "legs" of material, M-N-S-R etc. This results in an all-round horizontal compression of amount q in the pyramid-shape volume L-M-N-O-C. The uniaxial compression $2k$ is supported and carried down by the biaxial state of compression-tension of amount Q', q' respectively, in the four volumes of material, R-S-C-F etc. A simple tension of amount q' perpendicular to R-S in volume R-W-K-C (Fig. 3b, not shown in Fig. 2) and similarly a simple tension q' perpendicular to T-V in volume V-W-H-C are added horizontally outside the volumes R-S-C-F and T-V-C-F, respectively. The line C-W is then the intersection line of the volumes R-W-K-C and V-W-H-C. The stress system in region H-W-K-F-C is assumed to be composed of two equal tensions of amount q' acting in a horizontal direction, and region K-I-F-C or J-H-F-C is in a state of tension-tension-compression of amount q', q', and Q' respectively. For clarity, only two of the four supporting "legs" are shown in the diagram. Fig. 3a shows the plan view through the square area of punch, and Fig. 3b shows the plane section through the lines T-V and R-S. Sections by plane parallel to Fig. 3b, between the two points C and F, are similar in shape to the section shown in Fig. 3c, but are of varying size. Fig. 1 is then the vertical section through the mid-points of the two parallel sides of the square.

The three-dimensional pattern, as shown in Fig. 2 and Fig. 3, is a statically admissible stress field for the loading of a square punch on a square block. The value $Q = 2k\ (1 + \sin\alpha)$ is, therefore, a lower bound for the square punch on a square block for blocks whose width is greater than $\tan^2\ (\pi/4 + \alpha/2)$ times the width of the punch.

In order to increase the lower bound, a vertical compression of amount Q is superimposed in the volume vertically below the square area of contact, increasing the pressure on the area to $2k(1 + \sin\alpha) + Q$. In addition, a horizontal compression of amount Q is superimposed throughout the material in a cylindrical volume which circumscribes the square area of punch (Fig. 4).

The horizontal pressure Q in the cylinder is carried by a circular tube of interior radius $\sqrt{2}a$ and exterior radius r which is in a stress state whose circumferential stress component may be taken to be zero, and whose radial component is $\sqrt{2}aQ/r$ at the exterior face r so that the equilibrium conditions are satisfied for this tube. The second tube between the interior radius r and exterior radius b is in the state of fully plastic stress distribution, subjected to the internal pressure $\sqrt{2}aQ/r = 2k \ln b/r$ [See Ref. 6, for example]. Therefore, the average pressure on the square punch has the value $2k\ (1 + \sin\alpha) + Q$ which may be expressed as

$$p^{\ell} = 2k\ (\ 1 + \sin\alpha\ )$$
$$+ \sqrt{2}k\ \left[\ 1 + \tan^4\ (\tfrac{\pi}{4} + \tfrac{\alpha}{2})\ \right]^{1/2}\ \ln\left\{\frac{b}{a}\ \left[\ 1 + \tan^4\ (\tfrac{\pi}{4} + \tfrac{\alpha}{2})\ \right]^{-1/2}\right\} \qquad (1)$$

valid for $|Q| \leq 2k$.

The expression has a maximum value when α satisfies the condition $\partial p^{\ell}/\partial\alpha = 0$. This condition is

$$\ln\frac{b}{a} = 1 - \sqrt{2}\cos\alpha\ \left[\ 1 + \tan^4\ (\tfrac{\pi}{4} + \tfrac{\alpha}{2})\ \right]^{1/2}\ \cos^2\ (\tfrac{\pi}{4} + \tfrac{\alpha}{2})\ \cot^3\ (\tfrac{\pi}{4} + \tfrac{\alpha}{2})$$
$$+ \frac{1}{2}\ \ln\left[\ 1 + \tan^4\ (\tfrac{\pi}{4} + \tfrac{\alpha}{2})\ \right] \qquad (2)$$

It can be verified that Tresca's criterion is nowhere violated in the resulting stress field, so that the stress field is statically admissible and, by the Lower Bound Theorem of limit analysis[2], the value given in expression (1) is a lower bound for the collapse value of the average indentation pressure p. The values of p^{ℓ} are plotted against b/a in Fig. 5 (marked I).

The stress field in Fig. 2 can be modified, so that it applies to any area of contact which is convex. This follows directly from the fact that any convex area of contact can be closely approximated by an inscribed polygon. The associated stress fields below the area of contact, which may be visualized as supporting "legs" on each side of the polygon (in a manner essentially the same as shown in Fig. 2), do not overlap one another as long as the contact area is convex. In particular, the relevent formula for the average indentation pressure of a circular punch on a circular cylinder can be obtained directly from Fig. 4, by using an inner circular tube of interior radius a and exterior radius $r = a \tan^2 (\pi/4 + \alpha/2)$. The relevant lower bound formula is found to be given by

$$p^{\ell} = 2k \left\{ 1 + \sin\alpha + \tan^2 \left(\frac{\pi}{4} + \frac{\alpha}{2}\right) \ln \left[\frac{b}{a} \tan^2 \left(\frac{\pi}{4} - \frac{\alpha}{2}\right) \right] \right\} \qquad (3)$$

This expression has a maximum value when α satisfies the condition $\partial p^{\ell}/\partial\alpha = 0$, which yields

$$\ln \frac{b}{a} = 1 - \cos\alpha \cos^2 \left(\frac{\pi}{4} + \frac{\alpha}{2}\right) \cot \left(\frac{\pi}{4} + \frac{\alpha}{2}\right) + 2 \ln \tan \left(\frac{\pi}{4} + \frac{\alpha}{2}\right) \qquad (4)$$

Values of p^{ℓ} are plotted against b/a in Fig. 5 (marked II).

For a circular cylinder for which b/a is greater than 3.59 (rough punch) or 3.20 (smooth punch), the average indentation pressure becomes equal to that of a circular punch on the surface of a semi-infinite solid (6.05k for a rough punch or 5.69k for a smooth punch[3,5]). A lower bound for the indentation pressure of this limiting case by a square punch can be obtained by multiplying the circular case by $\pi/4$. This follows directly from the fact that the square area of contact can be inscribed by a circular area, so that the average indentation pressure over the square area is $\pi a^2/4a^2 = \pi/4$ times the lower bound value for a circular punch (the indentation pressure outside the inscribed circle is assumed zero).

These limiting values of the lower bound are also plotted against b/a in Fig. 5, and are joined by a smooth curve with some of the previously obtained corresponding points (marked III, IV, V, and VI. Curves III and V for rough punches and curves IV and VI for smooth punches).

3. SQUARE AND CIRCULAR PUNCHES - UPPER BOUNDS

An upper bound for the punch pressure in the indentation of a square punch on a square block can be obtained from the velocity field in which the volume A-B-E-C-D slides as a rigid body along the diagonal direction A-D (Fig. 6). Equating the rate of internal energy dissipation to the rate of external work yields the upper bound

$$p^u = k \left(1 + \frac{b}{a} \right)^2 \left(\frac{2 + \tan^2\beta}{4 \sin\beta} \right)^{1/2} \qquad (5)$$

The upper bound has a minimum value at the point $\beta = 50^\circ$, when it has the value

$$p^u = 0.604k \left(1 + \frac{b}{a} \right)^2 \qquad (6)$$

Thus, for a punch for which $b/a = 2$, expression (6) gives the value 5.42k, so that 5.42k is an upper bound for the collapse pressure. The almost straight line (dashed line) in Fig. 5 shows the values of p^u plotted against b/a*.

The velocity field in Fig. 6 may be modified to provide an upper bound for the collapse pressure in the circular punch problem. Only the plan view of the modified velocity field is shown in Fig. 7. The planes A-B-D and A-C-D are planes of velocity discontinuity. The volume A-B-E-C-D slides as a rigid body along the direction A-D in a similar manner

*For $b/a \leq 1.5$, the rigid body sliding along one side of the square punch gives upper bounds which are slightly less than those obtained by the rigid body sliding as shown in Fig. 6.

to that shown in Fig. 6. Application of the Upper Bound Theorem of limit analysis[2] gives

$$p^u = \frac{k}{\pi} \csc\beta \left[1 + \tan^2\beta \csc^2\delta \right]^{1/2} \left[\left(\frac{b^2}{a^2} - 1 \right)^{1/2} + \cot\delta \right.$$

$$\left. + \left(\delta + \sin^{-1}\frac{a}{b} \right) \frac{b^2}{a^2} \right] \tag{7}$$

The minimum value of the upper bound value p^u in Equation (7) is furnished by the simultaneous conditions

$$\frac{\partial p^u}{\partial \delta} = 0 \quad , \quad \frac{\partial p^u}{\partial \beta} = 0 \tag{8}$$

which leads to

$$\sin\delta = \tan^2\beta \tag{9a}$$

and

$$\left[1 + \sin\delta \right] \left[\frac{b}{a} \tan\delta - 2 \frac{a}{b} \csc 2\delta \right] - \frac{a}{b} \cot\delta - \frac{b}{a} \delta$$

$$= \frac{b}{a} \sin^{-1}\frac{a}{b} + \cos \left(\sin^{-1}\frac{a}{b}\right) \tag{9b}$$

The almost straight line (solid line) in Fig. 5 shows the values of p^u plotted against b/a.

There is no relative motion across the area of contact between the punch and block (Figs. 6 and 7), so that the upper bounds in expressions (5) and (7) are applicable to both rough and smooth punches.

4. CONCLUSION

Three-dimensional punch problems dealing with the indentation of a square punch on a square block, and a cricular punch on a circular cylinder are of considerable difficulty at the present stage of elastic-plastic analysis. However, as in the early work on the indentation of a punch on an infinitely large block[4], or in the more recent work on the indentation of concrete blocks and rock[1], the limit theorems of perfect plasticity[2] can be used to determine upper and lower bounds for the punch pressure during incipient plastic yield of the block.

Suppose the question is asked, what ratio b/a is needed in order to support the force $4k\,(\pi a^2)$ in the circular punch problem? The lower bound curve of Fig. 5 provides the information that 2.4 is sufficient, but will a smaller ratio be enough? There is no reason to suppose that the minimum value has been found, since the upper bound curve indicates that any ratio less than 1.5 is unsafe. This then suggests the use of average curve between upper and lower bound which is, in fact, quite close to a straight line (denoted by dash and small circle in Fig. 5). The average curve for the square punch is not shown, but may be determined in a similar manner to that described above.

5. ACKNOWLEDGMENTS

The research reported here was supported by the National Science Foundation under Grant Gk-3245 to Lehigh University.

The author would like to thank Professor T. Hirst for his review of the manuscript, and is indebted to Miss J. Arnold for typing the manuscript and to Mr. J. Gera for the drafting.

6. Figures

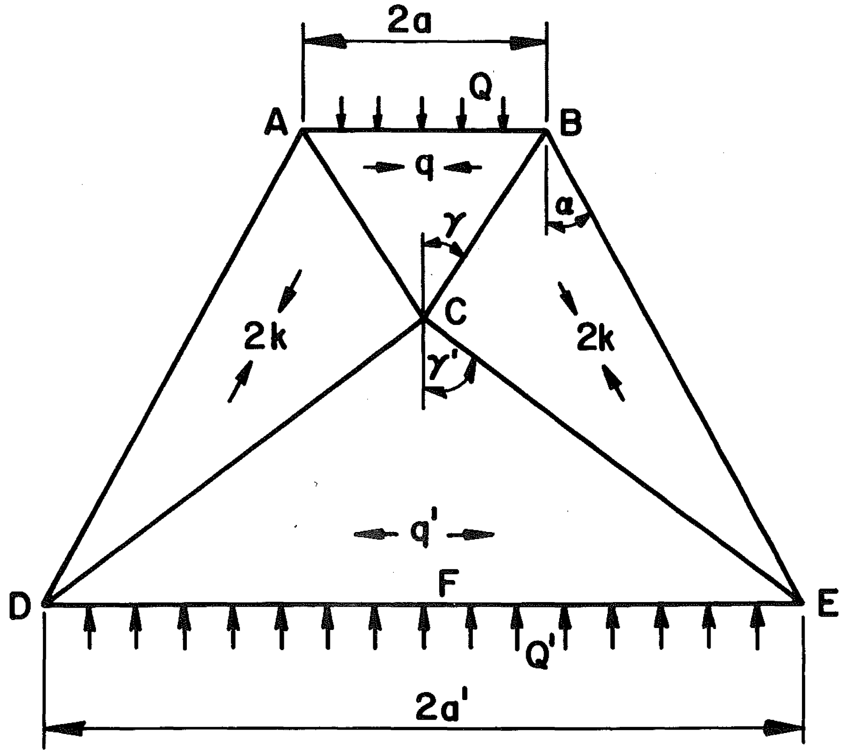


Fig. 1 Discontinuous Stress Field for Loaded Wedge

$$a' = a \tan^2 \left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$$

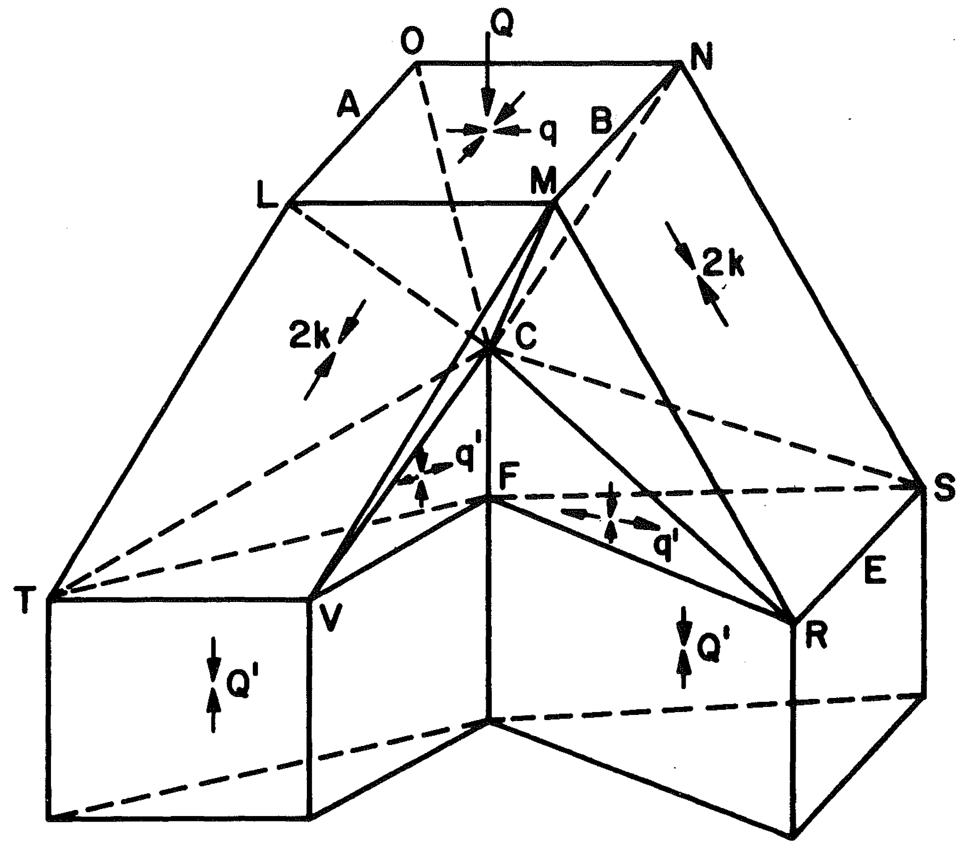


Fig. 2 Stress Field for Three-Dimensional Square Punch

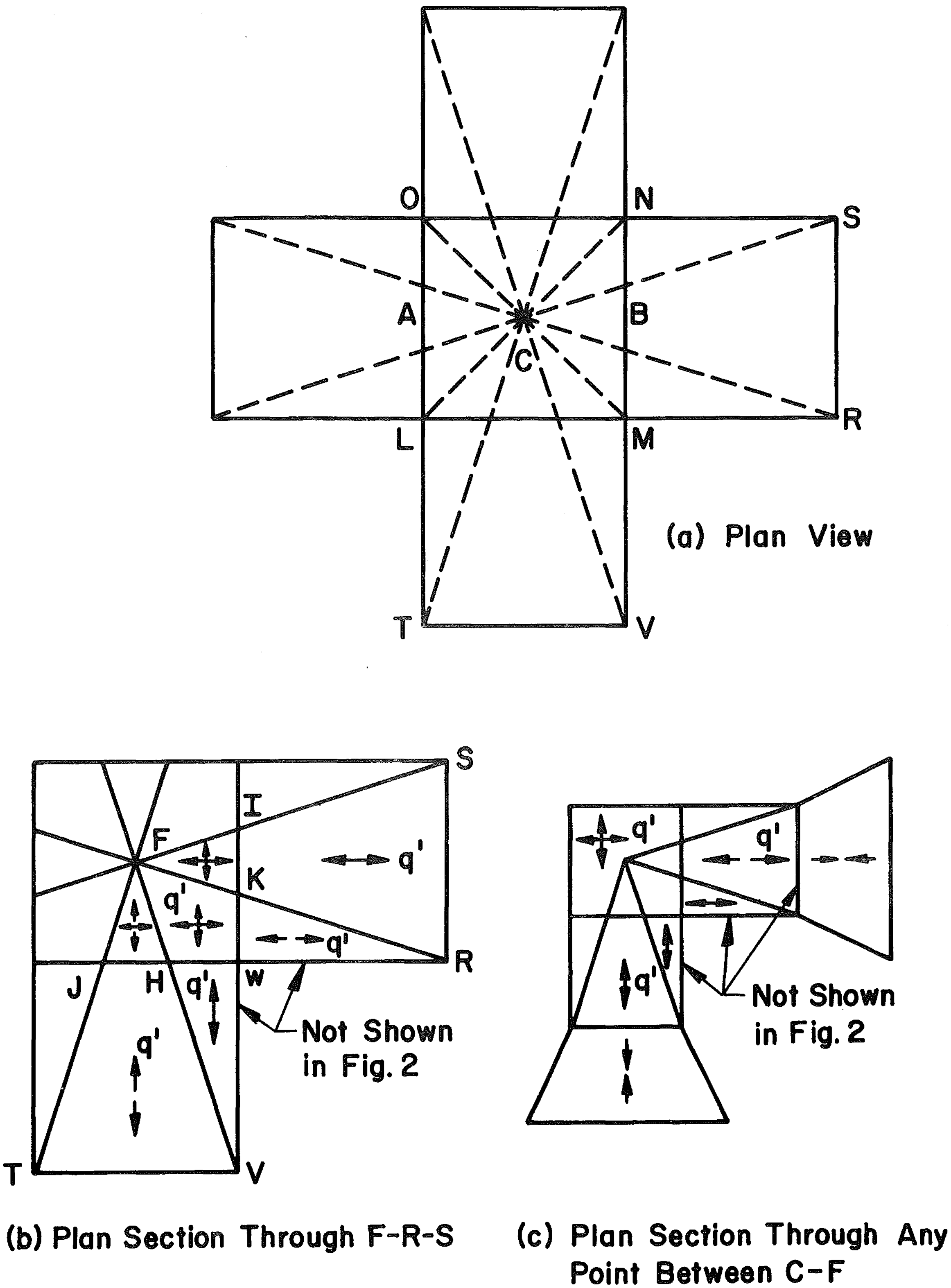
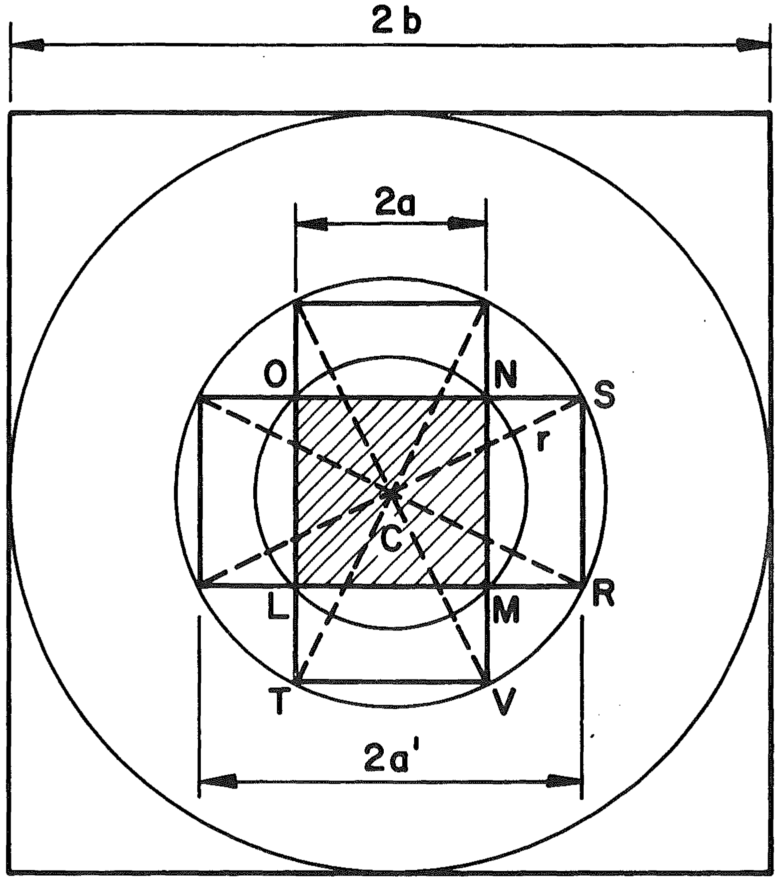


Fig. 3 Plan Sections in Fig. 2



$$r = a\left[1 + \tan^4\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right]^{1/2}$$

Fig. 4 Improvement of the Stress Field in Fig. 2

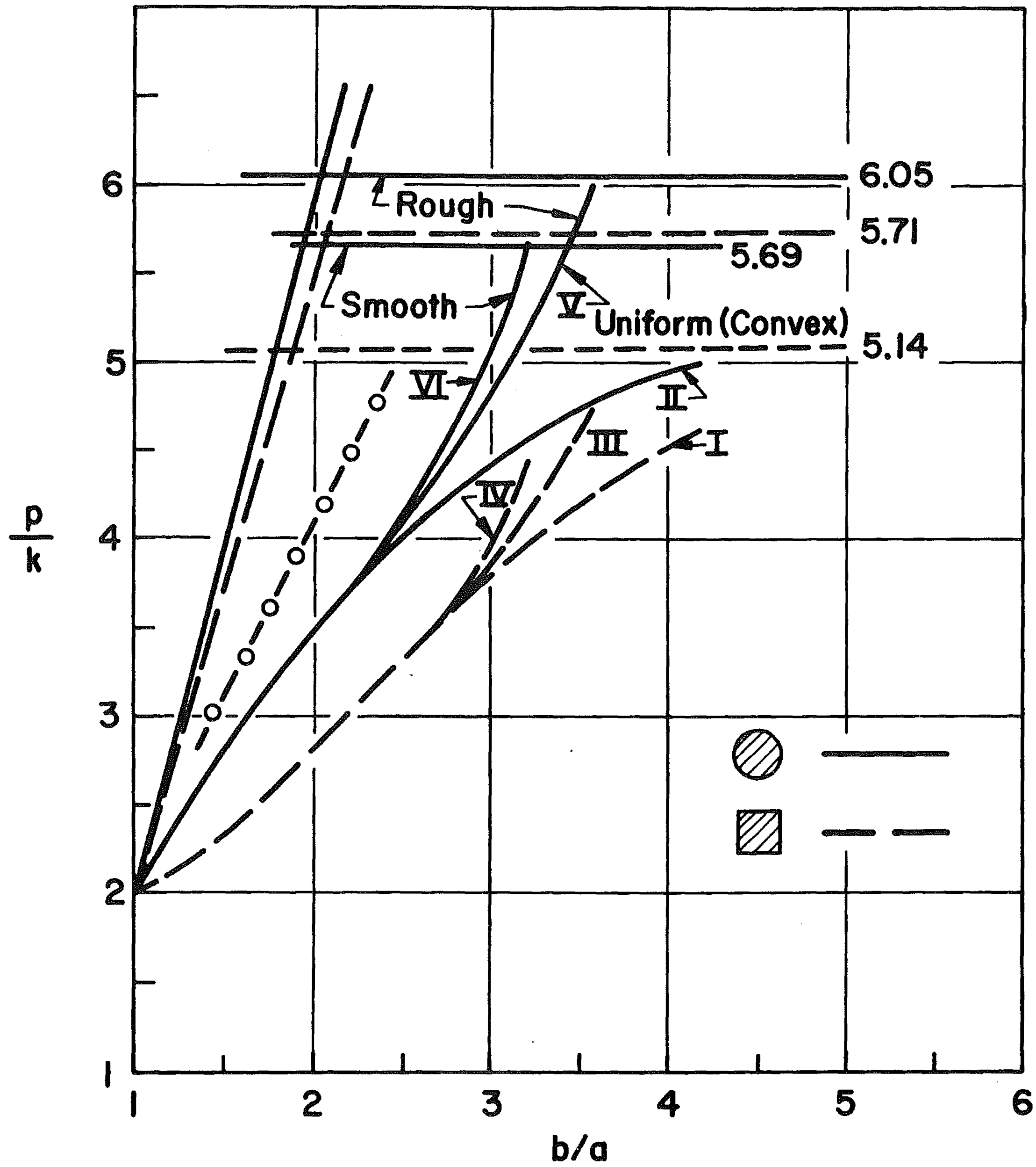


Fig. 5 Upper and Lower Bounds for a Square Punch on a Square Block and a Circular Punch on a Circular Cylinder

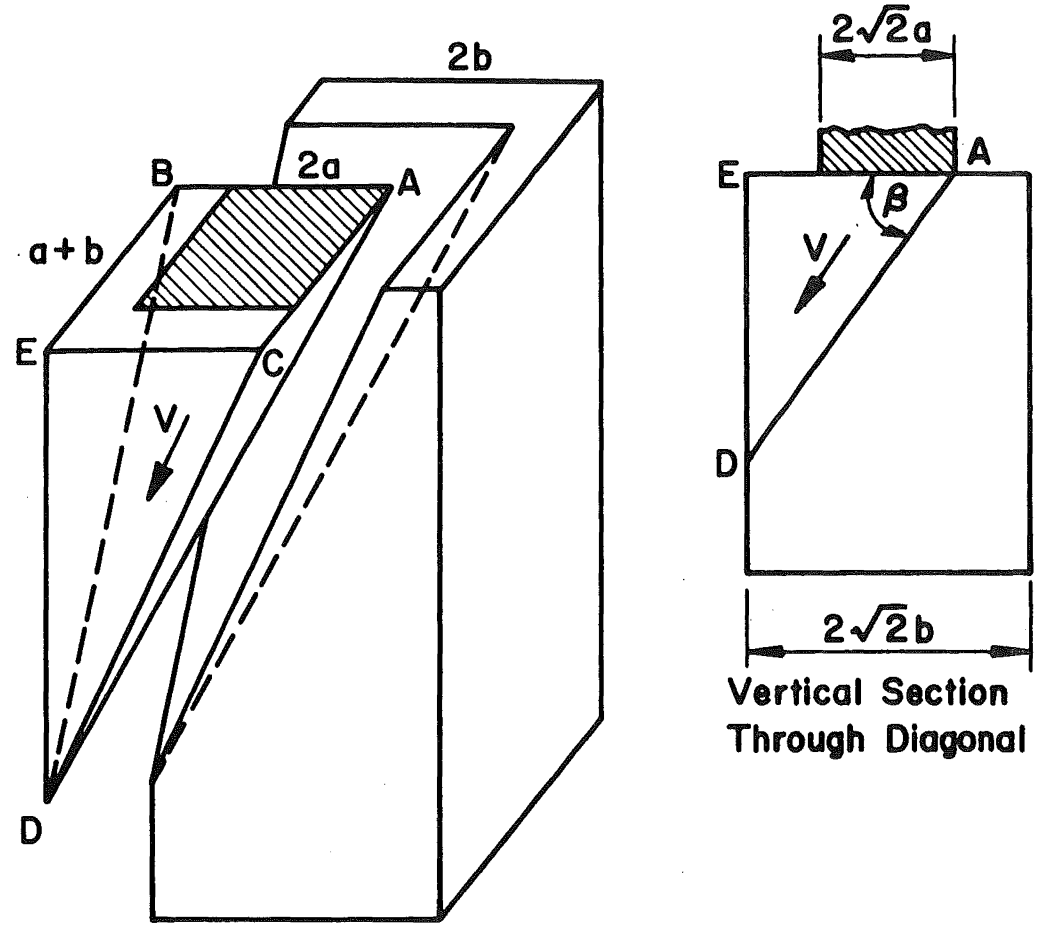


Fig. 6 Velocity Field for Square Punch on Square Block

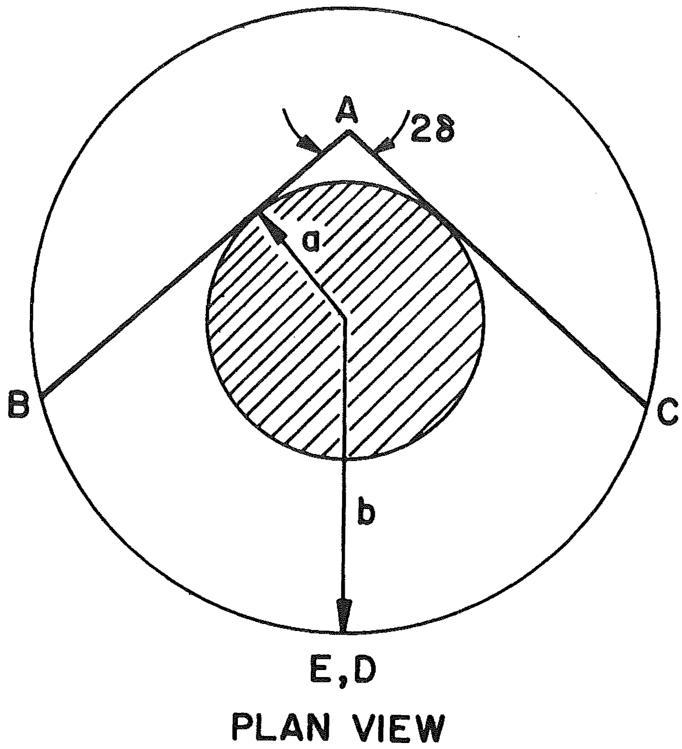


Fig. 7 Modification of Velocity Field of Fig. 6 to Circular Punch on Circular Cylinder

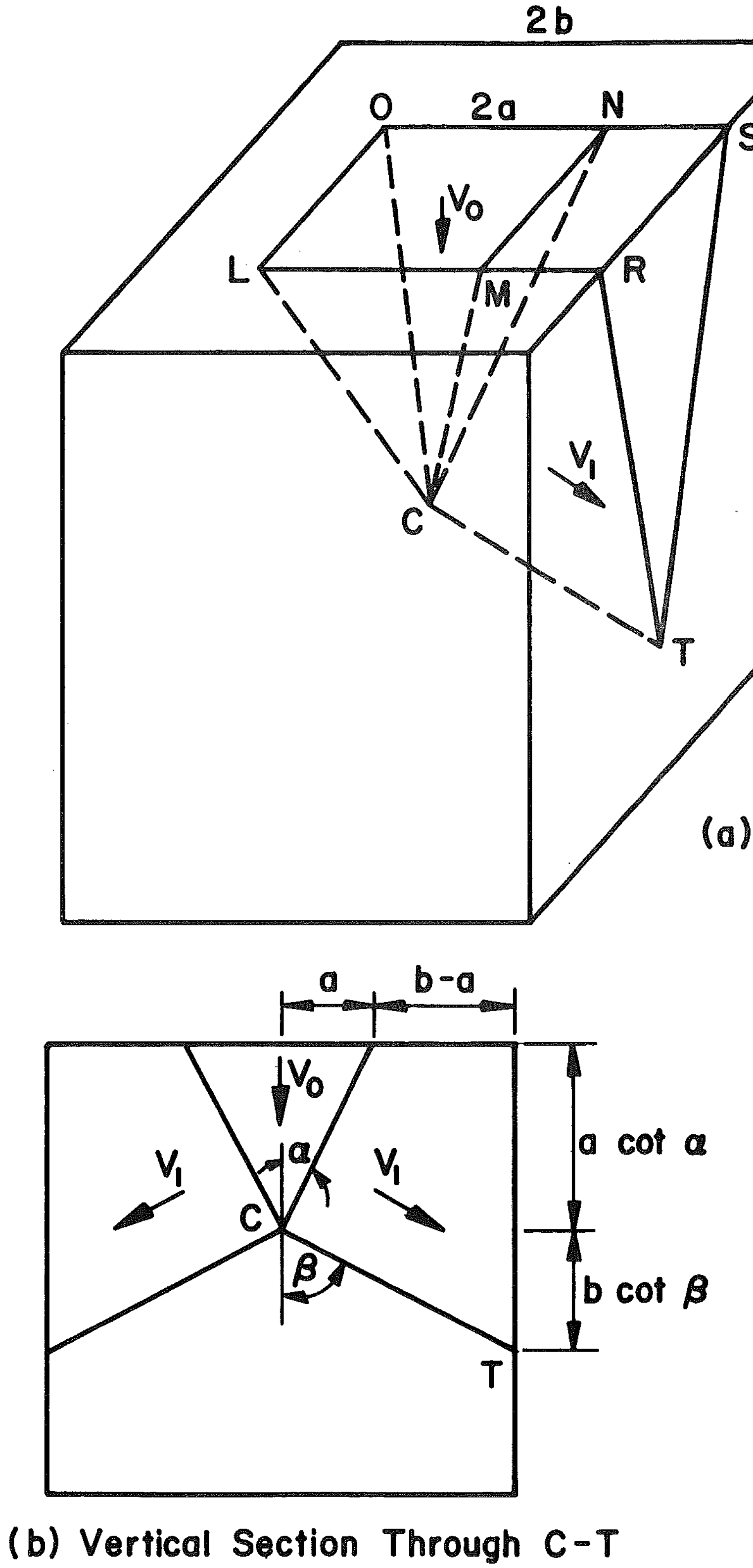


Fig. 8 Alternative Velocity Field for Square Punch

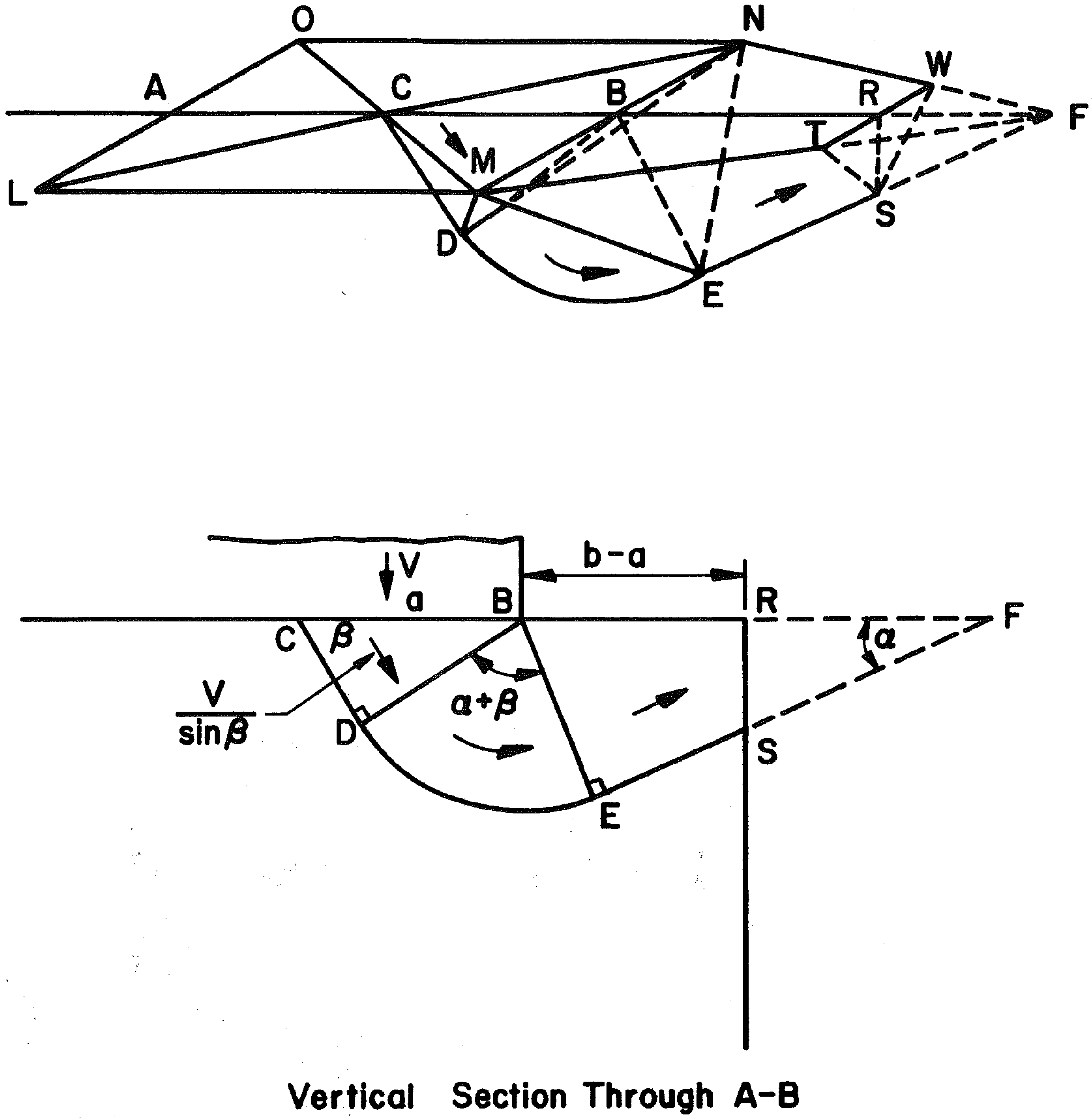


Fig. 9 Modification of the Velocity Field in Ref. 4 to Square Blocks

7. APPENDIX

Three-Dimensional Square Punch--Alternative Upper Bounds

(a) Upper Bound I

A simple discontinuous velocity field for the square punch is shown diagrmmmatically in Fig. 8. L-M-N-0 is the square area of the punch, and the initial downward velocity of this area is taken to be V_o. The downward movement of the pyramid-shape volume L-M-N-0-C is accommodated by movement of the four rigid volumes M-R-S-N-C-T, etc. as indicated in Fig. 8(a). For clarity, only one of the four volumes is shown in the diagram. The lines M-R and N-S are parallel to the sides L-M and 0-N of the square punch. Fig. 8(b) shows the vertical cross section through the mid-points of the two opposite sides of the square.

Since the velocity field is symmetrical about the two diagonal vertical planes passing through the opposite corners of the square, we need only to consider that part of the velocity field which supports the pressure on one-fourth of the punch. The internal dissipation of energy is due to the discontinuity surfaces between the material at rest, and the material in motion (surfaces M-R-T-C and N-S-T-C), and the discontinuity surface M-N-C where the two rigid materials have a relative velocity $V_o \sin\beta \,/\, \sin(\alpha+\beta)$.

The discontinuous surface M-R-T-C is of area:

$$\frac{a^2}{2} \left[\left(2\frac{b}{a} - 1 \right) \cot\alpha + \frac{b^2}{a^2} \cot\beta \right] \left[1 + \left(\cot\alpha + \frac{b}{a} \cot\beta \right)^{-2} \right]^{1/2} \tag{10}$$

and the triangle surface M-N-C is of area

$$\frac{1}{2}\,(2a)\,\left(\frac{a}{\sin\alpha}\right) \tag{11}$$

The rate of dissipation of energy is found by multiplying the area of each discontinuity surface by k times the discontinuity in velocity across the surface, and summing over all the surfaces.

The rate of external work is:

$$pa^2\ V_o \tag{12}$$

Equating the rate of total internal energy dissipation to the rate of external work yields

$$\frac{p^u}{k} = \frac{\sin\alpha}{\sin(\alpha+\beta)}\left[\left(2\,\frac{b}{a} - 1\right)\cot\alpha + \frac{b^2}{a^2}\cot\beta\right]\left[1 + \left(\cot\alpha + \frac{b}{a}\cot\beta\right)^{-2}\right]^{1/2} + \frac{\sin\beta}{\sin\alpha\,\sin(\alpha+\beta)} \tag{13}$$

With b/a = 2, the expression has the value 5.54k near the point $\alpha = 30^o$, $\beta = 70^o$. It may be noted that the expression is not sensitive to the variables α and β, so that 5.54k is almost the best upper bound for the collapse pressure.

(b) Upper Bound II

The velocity field used in Ref. 4 can be modified so that it can be used to provide upper bounds for the average limit pressure of the square punch problem. The diagram shown in Fig. 9 is essentially the same picture as given in Ref. 4 so that, in the interest of brevity, only the relevant expressions will be given here.

$$\text{Area M-T-S-E} = (\text{Area B-E-S-R}) \frac{a\left[1+\sin^2\beta\right]^{1/2}}{a \sin\beta}$$
$$= \frac{a^2}{2} \sin\beta \cot\alpha \left[1+\sin^{-2}\beta\right]^{1/2} f(\alpha,\beta) \tag{14}$$

where

$$f(\alpha,\beta) = 1 - \left[\tan\alpha \csc\beta \left(1 - \frac{b}{a}\right) + \sec\alpha\right]^2$$

$$0 \leq f \leq 1$$

$$\text{Area C-M-D} = \frac{1}{2} (a \cos\beta)(a \sin\beta) \frac{a\left[1+\sin^2\beta\right]^{1/2}}{a \sin\beta}$$
$$= \frac{1}{2} a^2 \cos\beta \left[1+\sin^2\beta\right]^{1/2} \tag{15}$$

$$\text{Area M-D-E} = \frac{1}{2} (\alpha+\beta)(a \sin\beta)^2 \frac{a\left[1+\sin^2\beta\right]^{1/2}}{a \sin\beta}$$
$$= \frac{a^2}{2} (\alpha+\beta) \sin\beta \left(1+\sin^2\beta\right)^{1/2} \tag{16}$$

Application of Upper Bound Theorem of limit analysis determines the upper bound

$$\frac{p^u}{k} = \alpha + \beta + \left(1+\sin^2\beta\right)^{1/2} \left[\alpha + \beta + \cot\beta + \cot\alpha\, f(\alpha,\beta)\right] \tag{17}$$

The function $f(\alpha,\beta)$ varies from 0 to 1. When $f = 0$; the angle α is then zero, and the lines N-W, M-T, and E-S are all parallel to the sides L-M and N-O of the square punch. When $f = 1$; the points F and R will be coincident, and the velocity field reduces to the field given in Ref. 4. For $f > 1$, the solution is independent of the ratio b/a, and the local solution, as given in Ref. 4, will govern. Here, the upper bound has the minimum value 5.80k when α and β are approximately 47° and 34°, respectively ($b/a \geq 1.767$).

It is found that the modified velocity field in Fig. 9 gives upper bounds which are higher than those obtained by the local solution 5.80k. For example, for $b/a = 2$, the Expression 17 has the minimum value 6.12k when α and β are approximately 20° and 60°, respectively. Expression 17 is found to be insensitive to the variables α and β, and thus the local solution 5.80k will be applicable for all blocks for which b/a is greater than 1.767.

8. NOMENCLATURE

a	punch width
b	block width
a'	bottom width of a truncated wedge stress field shown in Fig. 1
$f(\alpha,\beta)$	function defined in Equation (14)
k	maximum shearing stress permissible in the body
p	average punch pressure
q, q'	horizontal stress
Q, Q'	vertical pressure
r	radius, see Fig. 4
V, V_o, V_1	velocity
$\alpha, \beta, \gamma, \gamma', \delta$	angular parameter

9. REFERENCES

1. Chen, W. F. and Drucker, D. C.
THE BEARING CAPACITY OF CONCRETE BLOCKS OR ROCK, Brown University Report, Division of Engineering, Providence, Rhode Island, October, 1968.

2. Drucker, D. C., Greenberg, H. J., and Prager, W.
EXTENDED LIMIT DESIGN THEOREMS FOR CONTINUOUS MEDIA, Quarterly Applied Mathematics 9, pp. 381-389, (1952).

3. Eason, G. and Shield, R. T.
THE PLASTIC INDENTATION OF A SEMI-INFINITE SOLID BY A PERFECTLY ROUGH CIRCULAR PUNCH, Journal Applied Mathematical Physics (ZAMP), Vol. 11, pp. 33-43, 1960.

4. Shield, R. T. and Drucker, D. C.
THE APPLICATION OF LIMIT ANALYSIS TO PUNCH-INDENTATION PROBLEMS, Journal of Applied Mechanics, Trans. ASME, Vol. 75, pp. 453-460, 1953.

5. Shield, R. T.
ON THE PLASTIC FLOW OF METALS UNDER CONDITIONS OF AXIAL SYMMETRY, Proceedings of the Royal Society, A, Vol. 233, pp. 267-286, 1955.

6. Prager, W. and Hodge, P. G., Jr.
THEORY OF PERFECTLY PLASTIC SOLIDS, John Wiley & Sons, Inc., New York, New York, pp. 115-118, 1951.

7. Winzer, A. and Carrier, G. F.
THE INTERACTION OF DISCONTINUITY SURFACES IN PLASTIC FIELDS OF STRESS, Journal of Applied Mechanics, Trans. ASME, Vol. 70, p. 261, 1948.